PRESSURE OF A DIE ON AN ELASTIC LAYER OF FINITE THICKNESS

(O DAVLENII SHTAMPA NA SLOI KONECHNOI TOLSHCHINY)

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I.I. VOROVICH and Iu.A. USTINOV
(Rostov-on-Don)

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This article considers the problem of the pressure of an axisymmetric die on an elastic layer of finite thickness. Solution to this problem is obtained in terms of asymptotic series in powers of h^{-1}, where h is the dimensionless thickness of the layer. The cases of plane and parabolic dies are studied in detail. The solution obtained adequately described the state of stress when the thickness of layer is of the order of the diameter of the surface of contact.*

1. Statement of problem. Let an elastic layer of finite thickness h rest on a rigid frictionless foundation. An axially-symmetrical die, loaded by a force P along its axis of symmetry, acts on this layer (Fig. 1).

Let the surface of contact between the die and layer be defined in cylindrical coordinates by the equation $z = \phi(\rho)$. The boundary of this surface is assumed to be a circle of unit radius. The plane xy will be located on the rigid foundation, and the z-axis will be directed along the axis of the die.

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* After this article was submitted for printing, it came to the attention of the authors that another paper, written by Lebedev and Ufliand [1] considers the axially-symmetric problem of a plane circular die on a layer of finite thickness. The solution given in [1] will be applicable, generally speaking, for any dimensionless thickness h. However, its use involves the numerical evaluation of a certain Fredholm integral with subsequent numerical finding of quadratures. In this respect the asymptotic formulas presented in this work are important, since in numerous cases they give directly all the fundamental characteristics of the problem.

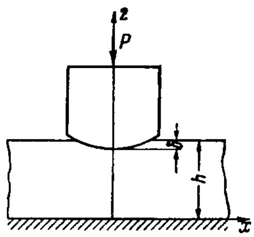

Fig. 1

Assume that the pressure between the die and layer is a known function $q(\rho)$. From [2], which considers equilibrium of an elastic layer subjected to the action of known forces on its boundary, we obtain the following relations:

$$w = W - \frac{1}{G}\int_0^\infty \frac{Q(\gamma)}{\Delta(\gamma h)}\, x_2(\gamma z)\, J_0(\gamma\rho)\, d\gamma \tag{1.1}$$

$$\tau_{\rho z} = T_{\rho z} + 2\int_0^\infty \frac{Q(\gamma)}{\Delta(\gamma h)}\, x_3(\gamma z)\, J_1(\gamma\rho)\,\gamma d\gamma \tag{1.2}$$

$$\sigma_z = \Sigma_z - 2\int_0^\infty \frac{Q(\gamma)}{\Delta(\gamma h)}\, x_4(yz)\, J_0(\gamma\rho)\,\gamma d\gamma \tag{1.3}$$

$$W = -\tfrac{1}{2}\Big\{(h-z)\int_0^\infty e^{-\gamma(h-z)}\, Q(\gamma) J_0(\gamma\rho)\gamma d\gamma - \tag{1,4}$$

$$-(h+z)\int_0^\infty e^{-\gamma(h+z)} Q(\gamma) J_0(\gamma\rho)\,\gamma d\gamma + \frac{2(m-1)}{m}\int_0^\infty [e^{-\gamma(h-z)} - e^{-\gamma(h-z)}] Q(\gamma)\, J_0(\gamma\rho) d\gamma\Big\}$$

$$T_{\rho z} = (h-z)\int_0^\infty Q(\gamma)\, e^{-\gamma(h-z)} J_1(\gamma\rho)\,\gamma^2\, d\gamma - (h+z)\int_0^\infty e^{-\gamma(h+z)} Q(\gamma)\, J_1(\gamma\rho)\,\gamma^2\, d\gamma \tag{1.5}$$

$$\Sigma_z = \int_0^\infty e^{-\gamma(h-z)}\, Q(\gamma)\, J_0(\gamma\rho)\,\gamma d\gamma - \int_0^\infty e^{-\gamma(h+z)}\, Q(\gamma)\, J_0(\gamma\rho)\,\gamma d\gamma - \tag{1.6}$$

$$-(h-z)\int_0^\infty e^{-\gamma(h-z)}\, Q(\gamma)\, J_0(\gamma\rho)\,\gamma^2 d\gamma - (h+z)\int_0^\infty e^{-\gamma(h+z)}\, Q(\gamma)\, J_0(\gamma\rho)\,\gamma^2 d\gamma$$

$$Q(\gamma) = \int_0^1 q(\rho)\, J_0(\gamma\rho)\,\rho d\rho, \qquad q(\rho) = 0, \qquad \text{if } \rho > 1 \tag{1.7}$$

$$\Delta(\gamma h) = 2\gamma h + \operatorname{sh} 2\gamma h \tag{1.8}$$

$$x_2(\gamma z) = e^{-\gamma h}\left[\gamma h a(\gamma h)\,\mathrm{sh}\,\gamma z + b(\gamma h)\left(\gamma z\,\mathrm{ch}\,\gamma z - \frac{2m-2}{m}\,\mathrm{sh}\,\gamma z\right)\right] \quad (1.9)$$

$$x_3(\gamma z) = e^{-\gamma h}\left[\gamma h a(\gamma h)\,\mathrm{sh}\,\gamma z + b(\gamma z)\,\gamma z\,\mathrm{ch}\,\gamma z\right] \quad (1.10)$$

$$x_4(\gamma z) = e^{-\gamma h}\left[\gamma h a(\gamma h)\,\mathrm{ch}\,\gamma z + b(\gamma h)(\gamma z\,\mathrm{sh}\,\gamma z - \mathrm{ch}\,\gamma z)\right] \quad (1.11)$$

$$a(\gamma h) = e^{-\gamma h}\,\mathrm{ch}\,\gamma h - 2\gamma h, \qquad b(\gamma h) = e^{-\gamma h}\,\mathrm{sh}\,\gamma h + 2\gamma h \quad (1.12)$$

For any function $Q(\gamma)$, formulas (1.1) to (1.6) satisfy the following boundary conditions:

$$w = \tau_{\rho z} = 0 \quad \text{for } z = 0, \qquad \tau_{\rho z} = 0 \quad \text{for } z = h \quad (1.13)$$

In order to fulfill two more conditions of contact between the layer and die, the function $Q(\gamma)$ must satisfy the following relations:

$$w = \delta - \varphi(\rho) \quad \text{for } 0 \leqslant \rho \leqslant 1, \quad z = h \quad (1.14)$$

$$\sigma_z = 0 \quad \text{for } 1 < \rho, \quad z = h \quad (1.15)$$

Here δ is the displacement of the die due to force P.

Considering relations (1.1), (1.4) and (1.7), we conclude that (1.14) and (1.15) will be satisfied if $Q(\gamma)$ is determined from equations:

$$\int_0^\infty \frac{\mathrm{ch}\,(2\gamma h) - 1}{\Delta(\gamma h)}\, Q(\gamma) J_0(\gamma\rho)\, d\gamma = c\,[\delta - \varphi(\rho)] \quad \left(c = \frac{m}{1-m}\right) \quad (0 \leqslant \rho \leqslant 1) \quad (1.16)$$

$$\int_0^\infty Q(\gamma) J_0(\gamma\rho)\, \gamma d\gamma = 0 \qquad (1 < \rho) \quad (1.17)$$

Thus the problem of die pressure on an elastic layer may be reduced to solving the paired integral equations (1.16) and (1.17).

2. Reduction of problem to Fredholm integrals. Method of solution. It is proved in [3] that formulas

$$\int_0^\infty Q(\gamma) J_0(\gamma\rho)\, d\gamma = g(\rho) \ (0 \leqslant \rho \leqslant 1), \qquad \int_0^\infty Q(\gamma) J_0(\gamma\rho)\, \gamma d\gamma = 0 \quad (1 < \rho) \quad (2.1)$$

are equivalent to

$$Q(\alpha) = \frac{2}{\pi}\cos\alpha \int_0^1 \frac{g(y)\,y}{\sqrt{1-y^2}}\, dy + \frac{2}{\pi}\int_0^1\int_0^1 \frac{y g(yu)\,\alpha u \sin \alpha u}{\sqrt{1-y^2}}\, dy\, du \quad (2.2)$$

provided that $q(\rho)$ is a continuous function on the interval [0,1].

For $Q(y)$ we obtain from (1.16) and (1.17)

$$\int_0^\infty Q(\gamma) J_0(\gamma\rho)\, d\gamma = c\,[\delta - \varphi(\rho)] + \int_0^\infty Q(\gamma) B(2\gamma h) J_0(\gamma\rho)\, d\gamma \qquad (0 \leqslant \rho \leqslant 1) \qquad (2.3)$$

$$\int_0^\infty Q(\gamma) J_0(\gamma\rho)\, \gamma d\gamma = 0 \qquad (1 < \rho) \qquad (2.4)$$

Here

$$B(2\gamma h) = \frac{1 + 2\gamma h - e^{-2\gamma h}}{2\gamma h + \operatorname{sh} 2\gamma h} \qquad (2.5)$$

Equating (2.1) - (2.5) we conclude that $Q(y)$ satisfies the following equation:

$$Q(\alpha) = \frac{2}{\pi} c \cos\alpha \int_0^1 \frac{y}{\sqrt{1-y^2}} [\delta - \varphi(y)]\, dy + \frac{2}{\pi} c \int_0^1\int_0^1 \frac{y\,[\delta - \varphi(yu)]}{\sqrt{1-y^2}} \alpha u \sin\alpha u\, dy\, du +$$
$$+ \frac{2}{\pi} \cos\alpha \int_0^1 \frac{y dy}{\sqrt{1-y^2}} \int_0^\infty Q(\gamma) B(2\gamma h) J_0(\gamma y)\, d\gamma +$$
$$+ \frac{2}{\pi} \int_0^\infty \int_0^1\int_0^1 \frac{y\alpha u \sin\alpha u}{\sqrt{1-y^2}} Q(\gamma) B(2\gamma h) J_0(\gamma y u)\, d\gamma\, dy\, dy \qquad (2.6)$$

Equation (2.6) may be simplified by utilizing the well-known relation

$$\int_0^1 \frac{J_0(uy)\, y}{\sqrt{1-y^2}} dy = \frac{\sin u}{u} \qquad (2.7)$$

From (2.7) the following formula is easily obtained:

$$\frac{2}{\pi} \cos\alpha \int_0^1 \frac{y dy}{\sqrt{1-y^2}} \int_0^\infty Q(\gamma) B(2\gamma h) J_0(\gamma\rho)\, d\gamma +$$
$$+ \frac{2}{\pi} \int_0^\infty \int_0^1\int_0^1 \frac{y\alpha u \sin\alpha u}{\sqrt{1-y^2}} Q(\gamma) B(2\gamma h) J_0(\gamma y u)\, d\gamma\, dy\, du =$$
$$= \frac{2}{\pi} \int_0^\infty \int_0^1 \cos\alpha u \cos\gamma u B(2\gamma h) Q(\gamma)\, d\gamma\, du \qquad (2.8)$$

In addition, it is clear that given the relation

$$\frac{2}{\pi} \cos\alpha \int_0^1 \frac{y dy}{\sqrt{1-y^2}} + \frac{2}{\pi} \int_0^1\int_0^1 \frac{y\alpha u \sin\alpha u}{\sqrt{1-y^2}} dy du = \frac{2}{\pi} \frac{\sin\alpha}{\alpha} \qquad (2.9)$$

(2.8) and (2.9) will simplify equation (2.6) to

$$Q(\alpha) = \frac{2}{\pi} c\delta \frac{\sin \alpha}{\alpha} - \frac{2}{\pi} c \left[\cos \alpha \int_0^1 \frac{y\varphi(y)}{\sqrt{1-y^2}} dy + \int_0^1\int_0^1 \frac{y\varphi(yu)\, \alpha u \sin \alpha u}{\sqrt{1-y^2}} dy\, du\right] +$$

$$+ \frac{2}{\pi} \int_0^\infty \int_0^1 \cos \alpha u \cos \gamma u B(2\gamma h) Q(\gamma)\, d\gamma\, du \qquad (2.10)$$

Let the operator $A(Q)$ be given by

$$A(Q) = \frac{2}{\pi} \int_0^\infty \int_0^1 \cos \alpha u \cos \gamma u B(2\gamma h) Q(\gamma)\, d\gamma\, du \qquad (2.11)$$

It is clear that A holds in the domain of functions $C[0, \infty]$, i.e. in the domain of continuous and bounded (on the semiaxis) functions with the norm

$$\| Q \| = \sup |Q| \qquad (0 \leqslant \alpha \leqslant \infty) \qquad (2.12)$$

We will show that A is a fully continuous operator.

Let $\{R\}$ be some multitude bounded in $C[0, \infty]$. We will establish that the multitude $\{A(R)\}$ is compact in this domain. To establish this it is sufficient to prove that $\{A(R)\}$ is uniformly bounded in $C[0, \infty]$ and uniformly continuous on $[0, \infty]$.

The first condition is clearly evident from (2.11) if we consider that

$$\int_0^\infty B(2\gamma h)\, d\gamma < \infty \qquad (2.13)$$

Further we have

$$|A[R(\alpha+\lambda)] - A[R(\alpha)]| \leqslant \frac{2}{\pi} \int_0^\infty \int_0^1 [\cos(\alpha+\lambda)u - \cos \alpha u] \times$$

$$\times |\cos \gamma u| B(2\gamma h) |R|\, d\gamma du \qquad (2.14)$$

From (2.14) we easily obtain (2.15)

$$|A[R(\alpha+\lambda)] - A[R(\alpha)]| \leqslant \frac{2}{\pi} \|R\| |\lambda| \int_0^\infty \int_0^1 u B(2\gamma h)\, d\gamma du = \frac{|\lambda| \|R\|}{\pi} \int_0^\infty B(2\gamma h) d\gamma$$

(2.13) and (2.15) establish the uniform continuity on $[0, \infty]$. Therefore this shows that A is a fully continuous operator in $C[0, \infty]$. Accordingly, in order to prove that equation (2.10) is a solution it must be shown that from the relation

$$D(\alpha) \equiv 0$$

where

$$D(\alpha) = \frac{2}{\pi} c\delta \frac{\sin \alpha}{\alpha} - \frac{2}{\pi} c \left[\cos\alpha \int_0^1 \frac{y\varphi(y)}{\sqrt{1-y^2}} dy + \int_0^1\int_0^1 \frac{y\varphi(uy)\,\alpha u \sin\alpha u}{\sqrt{1-y^2}} dydu\right] \quad (2.16)$$

it follows that $Q(\alpha) \equiv 0$ in view of equation (2.10).

We will prove first that if $D(\alpha) \equiv 0$ then it follows

$$\delta = 0, \qquad \varphi(\rho) \equiv 0 \quad (2.17)$$

Specifically, equation $D(\alpha) \equiv 0$ is equivalent to

$$\cos\alpha \int_0^1 \frac{[\delta - \varphi(y)]\,y}{\sqrt{1-y^2}} dy + \int_0^1\int_0^1 \frac{y\alpha u}{\sqrt{1-y^2}} [\delta - \varphi(y)] \sin\alpha u \, dudy = 0 \quad (2.18)$$

If $\alpha = 0$ is substituted into (2.18) we obtain

$$\int_0^1 \frac{y}{\sqrt{1-y^2}} [\delta - \varphi(y)]\, dy = 0 \quad (2.19)$$

As in the definition of δ on the surface of contact $\delta - \phi(\rho) \geqslant 0$, so from equation (2.19) we obtain $\delta - \phi(y) \equiv 0$ on the surface of contact.

Furthermore, if equation (2.10) had a non-trivial solution $Q^0(\alpha)$ for $D(\alpha) \equiv 0$, then formulas (1.1) to (1.6) with Q^0 substituted for Q, would always give the regular (at infinity) solution by the theory of elasticity for a layer with the boundary conditions given by (1.13), (1.15) and in addition with

$$w = 0, \quad \text{if} \quad 0 \leqslant \rho \leqslant 1 \quad (2.20)$$

Under these conditions $w = \tau_{\rho z} = \sigma_z = 0$ over the entire layer, which will result in $Q(\alpha) \equiv 0$. Therefore equation (2.10), which defines our problem, always has a unique solution.

Assuming that $\phi(y)$ is a twice continuously differentiable function on [0,1], we have

$$\frac{2}{\pi} c \left[\delta \frac{\sin\alpha}{\alpha} - \cos\alpha \int_0^1 \frac{y\varphi(y)}{\sqrt{1-y^2}} dy + \int_0^1\int_0^1 \frac{y\varphi(yu)\,\alpha u \sin\alpha u}{\sqrt{1-y^2}} dydu\right] = \quad (2.21)$$

$$= \frac{2}{\pi} c \left[\frac{\sin\alpha}{\alpha} \int_0^1 \frac{\delta y - \varphi(y)\, y - \varphi'(y)\, y^2}{\sqrt{1-y^2}} dy + \int_0^1\int_0^1 \frac{\sin\alpha u}{\alpha} \frac{2y^2\varphi'(yu) + uy^3\varphi''(yu)}{\sqrt{1-y^2}} dydu\right]$$

$$\int_0^\infty \int_0^1 \cos \alpha u \cos \gamma u B(2\gamma h) Q(\gamma)\, d\gamma\, du =$$
$$= \frac{\sin \alpha}{\alpha} \int_0^\infty B(2\gamma h) Q(\gamma) \cos\gamma\, d\gamma + \frac{1}{\alpha} \int_0^\infty \int_0^1 B(2\gamma h) Q(\gamma) \sin \alpha u \sin \gamma u\, d\gamma du \tag{2.22}$$

By means of (2.21) and (2.22) equation (2.10) may be written in the following form:

$$Q(\alpha) = \frac{2}{\pi} c \left[\frac{\sin \alpha}{\alpha} \int_0^1 \frac{\delta y - \varphi(y) y - \varphi'(y) y^2}{\sqrt{1-y^2}} dy + \right.$$
$$\left. + \int_0^1 \int_0^1 \frac{\sin \alpha u}{\alpha} \frac{2\varphi'(yu) + uy^3 \varphi''(yu)}{\sqrt{1-y^2}} dy du \right] + \frac{2}{\pi} \left[\frac{\sin \alpha}{\alpha} \int_0^\infty B(2\gamma h) Q(\gamma) \cos \gamma\, d\gamma + \right.$$
$$\left. + \frac{2}{\pi} \int_0^\infty \int_0^1 \gamma B(2\gamma h) Q(\gamma) \sin \alpha u \sin \gamma u\, d\gamma\, du \right] \tag{2.23}$$

On the right-hand side of (2.23) the Fourier-Bessel transforms in the first and third member will be irregular for $\rho = 1$. The Fourier-Bessel transforms in the second and fourth member will be regular over the entire plane. Consequently we will consider the following two cases:

I. The shape of the die is such that the surface of contact extends to the boundary of the die (Fig. 2). In this case the solution must be derived directly from equation (2.10).

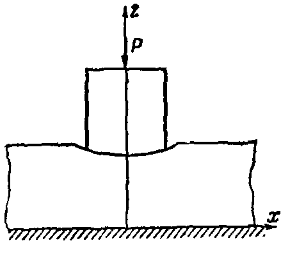

Fig. 2.

II. The surface of contact remains within the boundary of the die, in which case we must have

$$c \int_0^{+1} \frac{\delta y - y\varphi(y) - \varphi'(y) y^2}{\sqrt{1-y^2}} + \int_0^\infty B(2\gamma h) Q(\gamma) \cos \gamma\, d\gamma = 0 \tag{2.24}$$

and equation (2.23) becomes:

$$Q(\alpha) = \int_0^1\int_0^1 \frac{\sin\alpha u}{\alpha}\,\frac{2y^2\varphi'(yu) + uy^3\,\varphi''(yu)}{\sqrt{1-y^2}}\,dydu +$$

$$+\frac{2}{\pi}\int_0^\infty\int_0^1 \gamma B(2\gamma h)\,Q(\gamma)\sin\alpha u\sin\gamma u\,d\gamma du \qquad (2.25)$$

Solving(2.25) for Q we can obtain $q(\rho)$ and the corresponding force P from

$$q(\rho) = \int_0^\infty Q(\gamma)\,J_0(\gamma\rho)\,\gamma\,d\gamma \qquad (2.26)$$

From (2.24) we then obtain δ, the displacement of the die .

3. Solution of problem. Equation (2.10) may be written in the following form:

$$Q(\alpha) = Q_0(\alpha) + A(Q) \qquad (3.1)$$

Here $A(Q)$ is given by (2.11), and

$$Q_0(\alpha) = \frac{2}{\pi}c\delta\frac{\sin\alpha}{\alpha} - \frac{2}{\pi}c\left[\cos\alpha\int_0^1\frac{y\varphi(y)}{\sqrt{1-y^2}}dy + \int_0^1\int_0^1\frac{y\varphi(y)\,\alpha u\sin\alpha u}{\sqrt{1-y^2}}\,dydu\right] \qquad (3.2)$$

It is clear that $A(Q)$ is a convergent operator in the domain $C[0, \infty]$, if h is large enough. In fact, from (2.11) we obtain

$$\|A\| \leqslant \frac{2}{\pi}\int_0^\infty B(2\gamma h)\,d\gamma \qquad (3.3)$$

From (3.3) it can be shown by means of simple calculations that $A(Q)$ will be a convergent operator if $h > 4/\pi = 1.27$. As will be shown below, equation (3.1) can be easily solved by the method of successive approximations already when $h \approx 2$.

Similarly, equation (2.25) may be written in the following form:

$$Q(\alpha) = Q_0(\alpha) + K(Q) \qquad (3.4)$$

where

$$Q_0 = \int_0^1\int_0^1\frac{\sin\alpha u}{\alpha}\,\frac{2y^2\varphi'(yu) + uy^3\varphi''(yu)}{\sqrt{1-y^2}}\,dydu \qquad (3.5)$$

$$K(Q) = \frac{1}{\alpha}\int_0^\infty\int_0^1 \gamma B(2\gamma h)\,Q(\gamma)\sin\alpha u\,\sin\gamma u\,dud\gamma \qquad (3.6)$$

Simple numerical calculations using (3.6) show that $K(Q)$ will also be a convergent operator when $h > 0.97$. We will show later that (3.4) may be easily solved by successive approximations already when $h \approx 1.5$.

4. Plane cylindrical die. In this case

$$\varphi(\rho) = 0 \tag{4.1}$$

and the solution to the problem should be based on equation (2.10), which assumes the following form in view of (4.1):

$$Q(\alpha) = \frac{2}{\pi}\delta c\frac{\sin\alpha}{\alpha} + \frac{2}{\pi}\int_0^\infty\int_0^1 \cos\alpha u \cos\gamma u B(2\gamma h) Q(\gamma)\,d\gamma du \tag{4.2}$$

Solving (4.2) by successive approximations yields:

$$Q_0(\alpha) = \frac{2}{\pi}c\delta\frac{\sin\alpha}{\alpha} \tag{4.3}$$

$$Q_1(\alpha) = Q_0(\alpha) + \frac{2\delta c}{\pi^2 h}\int_0^1\int_0^\infty \cos\alpha y \cos\frac{yu}{2h} B(u)\frac{\sin(u/2h)}{u/2h}dudy \tag{4.4}$$

$$Q_2(\alpha) = Q_1(\alpha) + \frac{2\delta c}{\pi^3 h^2}\int_0^1\int_0^\infty\int_0^1\int_0^\infty \cos\alpha y \cos\frac{yu}{2h}\cos\frac{vx}{2h}\cos\frac{ux}{2h}\times$$

$$\times B(u)B(v)\frac{\sin(v/2h)}{v/2h}\,du\,dv\,dy\,dx \tag{4.5}$$

etc. Passing from $Q(\alpha)$ to $q(\rho)$ by means of (2.26) the following successive approximations are obtained for $q(\rho)$:

$$q_0(\rho) = \frac{2c\delta}{\pi\sqrt{1-\rho^2}} \tag{4.6}$$

$$q_1(\rho) = q_0(\rho) + \frac{2c\delta}{\pi^2 h\sqrt{1-\rho^2}}\int_0^\infty B(u)\frac{\sin(u/2h)}{u/2h}\cos\frac{u}{2h}du + \tag{4.7}$$

$$+\frac{2c\delta}{\pi^2 h}\int_\rho^1\frac{dy}{\sqrt{y^2-\rho^2}}\int_0^\infty B(u)\sin\frac{u}{2h}\sin\frac{yu}{2h}du$$

$$q_2(\rho) = q_1(\rho) + \tag{4.8}$$

$$+\frac{2c\delta}{\pi^3 h^2\sqrt{1-\rho^2}}\int_0^\infty B(u)\cos\frac{u}{2h}du\int_0^1\cos\frac{ux}{2h}dx\int_0^\infty B(v)\frac{\sin(v/2h)}{v/2h}\cos\frac{xv}{2h}dv +$$

$$+\frac{2\delta c}{\pi^3 h^2}\int_\rho^1\frac{dy}{\sqrt{y^2-\rho^2}}\int_0^\infty B(u)\frac{u}{2h}\sin\frac{yu}{2h}du\int_0^1\cos\frac{ux}{2h}dx\int_0^\infty B(v)\frac{\sin(v/2h)}{v/2h}\cos\frac{xv}{2h}dv$$

$$q_3(\rho) = q_2(\rho) + \frac{2c\delta}{\pi^4 h^3 \sqrt{1-\rho^2}} \int_0^\infty B(u) \cos\frac{u}{2h}\, du \int_0^1 \cos\frac{ux}{2h}\, dx \times$$
$$\times \int_0^\infty B(v) \cos\frac{xv}{2h}\, dv \int_0^1 \cos\frac{vz}{2h}\, dz \int_0^\infty B(\gamma)\, \frac{\sin(\gamma/2h)}{\gamma/2h} \cos\frac{\gamma z}{2h}\, d\gamma +$$
$$+ \frac{2c\delta}{\pi^4 h^3} \int_\rho^1 \frac{dy}{\sqrt{y^2-\rho^2}} \int_0^\infty B(u)\, \frac{u}{2h} \sin\frac{uy}{2h}\, du \int_0^1 \cos\frac{ux}{2h}\, dx \int_0^\infty B(v) \times$$
$$\times \cos\frac{vx}{2h}\, dv \int_0^1 \cos\frac{vz}{2h}\, dv \int_0^\infty B(\gamma)\, \frac{\sin(\gamma/2h)}{\gamma/2h} \cos\frac{z\gamma}{2h}\, d\gamma \qquad (4.9)$$

To simplify their use, (4.6) to (4.9) may be expressed in an asymptotic form for large values of h. This leads to the following approximate formulas:

$$q_0(\rho) = \frac{2c\delta}{\pi\sqrt{1-\rho^2}} \qquad (4.10)$$

$$q_1(\rho) = \frac{2c\delta}{\pi\sqrt{1-\rho^2}}\Big[1 + \frac{0.755}{h} + \frac{0.337}{h^3} + \frac{0.685}{h^5} - \rho^2\Big(\frac{1.012}{h^3} + \frac{3.42}{h^5}\Big) + \rho^4\,\frac{3.42}{h^5}\Big] \qquad (4.11)$$

$$q_2(\rho) = \frac{2c\delta}{\pi\sqrt{1-\rho^2}}\Big[1 + \frac{0.755}{h} + \frac{0.570}{h^2} + \frac{0.337}{h^3} + \frac{0.685}{h^5} - \frac{0.237}{h^6} - \rho^2\Big(\frac{1.012}{h^3} + \frac{0.763}{h^4} + \frac{3.42}{h^5} - \frac{0.770}{h^6}\Big) + \rho^4\Big(\frac{3.42}{h^5} + \frac{0.430}{h^6}\Big) + \cdots\Big] \qquad (4.12)$$

$$q_3(\rho) = \frac{2c\delta}{\pi\sqrt{1-\rho^2}}\Big[1 + \frac{0.755}{h} + \frac{0.570}{h^2} + \frac{0.767}{h^3} + \frac{0.492}{h^5} - \frac{0.237}{h^6} - \rho^2\Big(\frac{1.012}{h^3} + \frac{0.763}{h^4} + \frac{3.996}{h^5} - \frac{0.770}{h^6}\Big) + \rho^4\Big(\frac{3.42}{h^5} + \frac{0.430}{h^6}\Big) + \cdots\Big] \qquad (4.13)$$

$$q_4(\rho) = \frac{2c\delta}{\pi\sqrt{1-\rho^2}}\Big[1 + \frac{0.755}{h} + \frac{0.570}{h^2} + \frac{0.767}{h^3} + \frac{0.325}{h^4} + \frac{0.492}{h^5} - \frac{0.527}{h^6} - \rho^2\Big(\frac{1.012}{h^3} + \frac{0.763}{h^4} + \frac{3.996}{h^5} - \frac{0.335}{h^6}\Big) + \rho^4\Big(\frac{3.42}{h^5} + \frac{0.430}{h^6}\Big) + \cdots\Big] \qquad (4.14)$$

$$q_5(\rho) = \frac{2c\delta}{\pi\sqrt{1-\rho^2}}\Big[1 + \frac{0.755}{h} + \frac{0.570}{h^2} + \frac{0.767}{h^3} + \frac{0.325}{h^4} + \frac{0.737}{h^5} - \frac{0.527}{h^6} - \rho^2\Big(\frac{1.012}{h^3} + \frac{0.763}{h^4} + \frac{3.996}{h^5} - \frac{0.335}{h^6}\Big) + \rho^4\Big(\frac{3.42}{h^5} + \frac{0.430}{h^6}\Big) + \cdots\Big] \qquad (4.15)$$

$$q_6(\rho) = \frac{2c\delta}{\pi\sqrt{1-\rho^2}}\Big[1 + \frac{0.755}{h} + \frac{0.570}{h^2} + \frac{0.767}{h^3} + \frac{0.325}{h^4} + \frac{0.737}{h^5} - \frac{0.342}{h^6} + \rho^2\Big(-\frac{1.012}{h^3} - \frac{0.763}{h^4} - \frac{3.996}{h^5} + \frac{0.335}{h^6}\Big) + \rho^4\Big(\frac{3.42}{h^5} + \frac{0.430}{h^6}\Big) + \cdots\Big] \qquad (4.16)$$

Examination of formulas (4.10) to (4.16) indicates that in transition from asymptotic expansion q_n to asymptotic expansion q_{n+1} the coefficients of $h^{-k}(k = 0, 1, \ldots, n)$ remain identical. This leads us to conclude that the first n terms in the asymptotic expansion q_n coincide exactly with the first n terms in the expansion of q in asymptotic series.

In accordance with formulas (4.10) to (4.16), the following approximations are obtained for the force P acting on the die:

$$P_0 = 4c\delta \tag{4.17}$$

$$P_1 = 4c\delta\left[1 + \frac{0.755}{h} - \frac{0.337}{h^3} + \frac{0.228}{h^5} + \cdots\right] \tag{4.18}$$

$$P_2 = 4c\delta\left[1 + \frac{0.755}{h} + \frac{0.570}{h^2} - \frac{0.337}{h^3} - \frac{0.508}{h^4} + \frac{0.228}{h^5} + \frac{0.506}{h^6} + \cdots\right] \tag{4.19}$$

$$P_3 = 4c\delta\left[1 + \frac{0.755}{h} + \frac{0.570}{h^2} + \frac{0.093}{h^3} - \frac{0.508}{h^4} - \frac{0.348}{h^5} + \frac{0.506}{h^6} + \cdots\right] \tag{4.20}$$

$$P_4 = 4c\delta\left[1 + \frac{0.755}{h} + \frac{0.570}{h^2} + \frac{0.093}{h^3} - \frac{0.183}{h^4} - \frac{0.348}{h^5} - \frac{0.074}{h^6} + \cdots\right] \tag{4.21}$$

$$P_5 = 4c\delta\left[1 + \frac{0.755}{h} + \frac{0.570}{h^2} + \frac{0.093}{h^3} - \frac{0.183}{h^4} - \frac{0.104}{h^5} - \frac{0.074}{h^6} + \cdots\right] \tag{4.22}$$

$$P_6 = 4c\delta\left[1 + \frac{0.755}{h} + \frac{0.570}{h^2} + \frac{0.093}{h^3} - \frac{0.183}{h^4} - \frac{0.104}{h^5} - \frac{0.110}{h^6} + \cdots\right] \tag{4.23}$$

The accuracy of expansions (4.10) to (4.16) and (4.17) to (4.23) improves with increasing values of h. For example, for $h = 1.5$ the difference between fifth and sixth approximation of P is only 2 per cent, which indicates that formula (4.23) gives good approximation for $h > 1.5$.

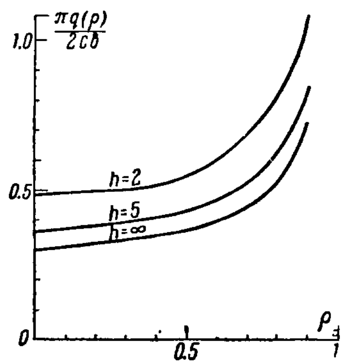

Fig. 3.

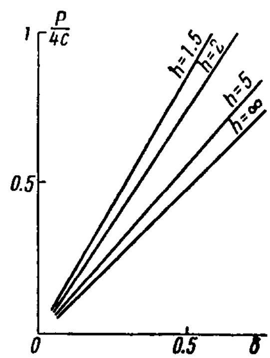

Fig. 4.

Similar analysis of formulas (4.10) to (4.16) shows that for $h = 1.7$ the difference between fifth and sixth approximations of the stress under the die is again 2 per cent. Therefore formula (4.15) gives good approximation for $h > 1.7$.

On the basis of formulas (4.10) to (4.16) and (4.17) to (4.23) we may analyze the effect of thickness of the layer on the stress distribution due to the loaded die.

Fig. 3 shows the variation of $q(\rho, h)$ given by formula (4.15). Fig. 4 illustrates the force of penetration P for different values of h.

5. Parabolic die. Let us now assume that $\phi(\rho)$ is given by

$$\varphi(\rho) = \frac{\rho^2}{2R} \tag{5.1}$$

Since the surface $\phi(\rho)$ is smooth in this case, equation (2.25) must be used to determine Q. Substituting (5.1) into (2.25) we obtain

$$Q(\alpha) = -\frac{4c}{\pi R}\frac{1}{\alpha}\frac{d}{d\alpha}\frac{\sin\alpha}{\alpha} + \frac{1}{\pi h\alpha}\int_0^1\int_0^\infty B(u)\,Q\left(\frac{u}{2h}\right)\frac{u}{2h}\sin\frac{uy}{2h}\sin\alpha y\,dy\,du \tag{5.2}$$

In this case formula (2.24) yields:

$$c\left(\delta - \frac{1}{R}\right) + \frac{1}{2h}\int_0^\infty Q\left(\frac{u}{2h}\right)B(u)\cos\frac{u}{2h}\,du = 0 \tag{5.3}$$

Recall that the boundary radius of the surface of contact was assumed as unity.

The results of successive approximations for the pressure under the die $q(\rho)$ are the following:

$$\begin{aligned} q_1(\rho) = \frac{4c}{\pi R}\sqrt{1-\rho^2}\Big[1 + \frac{0.337}{h^3} - \frac{0.266}{h^5} + \frac{0.024}{h^7} + \\ + \rho^2\left(-\frac{0.190}{h^5} + \frac{0.029}{h^7}\right) + \rho^4\frac{0.010}{h^7} + \cdots\Big] \end{aligned} \tag{5.4}$$

$$\begin{aligned} q_2(\rho) = \frac{4c}{\pi R}\sqrt{1-\rho^2}\Big[1 + \frac{0.337}{h^3} - \frac{0.266}{h^5} + \frac{0.114}{h^6} + \frac{0.024}{h^7} - \frac{0.205}{h^8} + \\ + \rho^2\left(-\frac{0.190}{h^5} + \frac{0.029}{h^7} - \frac{0.064}{h^8}\right) + \rho^4\frac{0.010}{h^7} + \cdots\Big] \end{aligned} \tag{5.5}$$

It can be easily recognized that further approximations will contain corrective terms of the order of h^{-9} and higher.

For the force P which will result in penetration δ consistent with the given loading conditions we obtain the following relations:

$$P_1 = \frac{8c}{3R}\left[1 + \frac{0.337}{h^3} - \frac{0.342}{h^5} + \frac{0.037}{h^7} + \cdots\right] \tag{5.6}$$

$$P_2 = \frac{8c}{3R}\left[1 + \frac{0.337}{h^3} - \frac{0.342}{h^5} + \frac{0.114}{h^6} + \frac{0.037}{h^7} - \frac{0.230}{h^8} + \cdots\right] \tag{5.7}$$

Formulas (5.4) to (5.7) are recommended when $h > 1.5$.

If we omit the requirement that the boundary radius of contact surface be unity, formula (5.7) assumes the form:

$$P_2 = \frac{8ca^3}{3R}\left[1 + 0.337\left(\frac{a}{H}\right)^3 - 0.342\left(\frac{a}{H}\right)^5 + 0.114\left(\frac{a}{H}\right)^6 + \right.$$
$$\left. + 0.037\left(\frac{a}{H}\right)^7 - 0.230\left(\frac{a}{H}\right)^8 + \cdots\right] \quad (5.8)$$

where a is the radius of the boundary of contact surface, and H is the thickness of the layer.

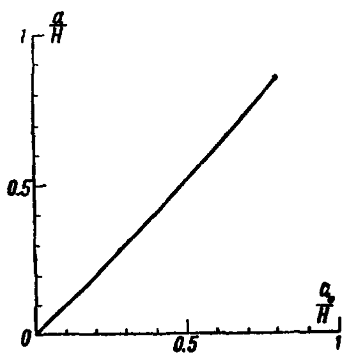

Fig. 5.

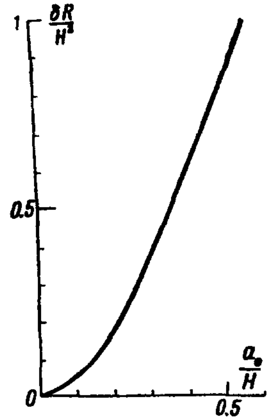

Fig. 6.

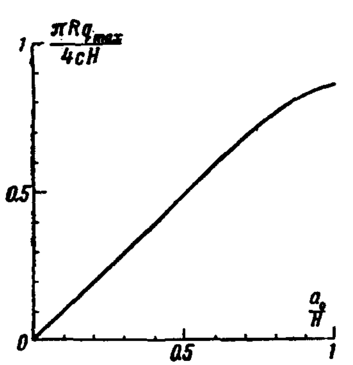

Fig. 7.

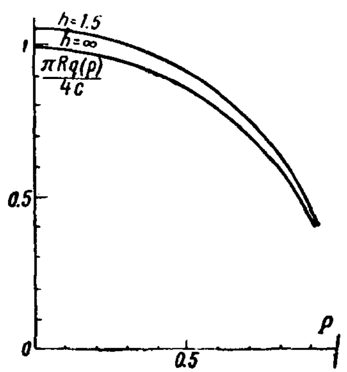

Fig. 8.

From (5.3), (5.5), and (5.8) the following formulas can be obtained:

$$\frac{a}{H}=\frac{a_0}{H}-0.113\left(\frac{a_0}{H}\right)^4+0.114\left(\frac{a_0}{H}\right)^6+0.025\left(\frac{a_0}{H}\right)^7-0.004\left(\frac{a_0}{H}\right)^8+\ldots \quad (5.9)$$

$$\delta=\frac{H^2}{R}\left[\left(\frac{a_0}{H}\right)^2-0.504\left(\frac{a_0}{H}\right)^3-0.225\left(\frac{a_0}{H}\right)^5-0.098\left(\frac{a_0}{H}\right)^7-0.197\left(\frac{a_0}{H}\right)^8+\ldots\right]$$

$$q_{\max}=\frac{4cH}{\pi R}\left[\frac{a_0}{H}+0.225\left(\frac{a_0}{H}\right)^4-0.018\left(\frac{a_0}{H}\right)^6-0.0126\left(\frac{a_0}{H}\right)^7+0.013\left(\frac{a_0}{H}\right)^8+\ldots\right]$$

$$a_0=\left[\frac{3PR(m-1)}{8m}\right]^{1/3}$$

Here q_{max} is the maximum pressure under the die. Figs. 5, 6, 7 illustrate the dependence given by (5.9), and Fig. 8 shows the dependence of $q(\rho)$ on the thickness h.

In conclusion it should be noted that equations (4.2) and (5.2), which define $Q(a)$ and which were shown to be Fredholm integrals, permit numerical integration for any value of h.

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